LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc., DEGREE EXAMINATION - **MATHEMATICS**

FIRST SEMESTER – NOVERMBER 2011

MT 1810/MT 1804 – LINEAR ALGEBRA

Date : 01/11/11 Dept. No. Max. : 100 Marks

Time: 1.00 - 4.00

I. a. i) Let $A=\begin{pmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{pmatrix}$ be the matrix of a linear operator T defined on R^3 with respect to the standard ordered basis. Prove that A is diagonalizable.

(OR)

ii) Prove that the similar matrices have the same characteristic polynomial. (5)

b. (i) Let T be a linear operator on finite dimensional space V and $c_1, \dots c_k$ be the distinct characteristic values of T. Let W_i be the null space of $(T - c_iI)$. Prove that the following are equivalent.

1. T is diagonalizable
2. The characteristic polynomial for T is $f=(x-c_1)^{d1}...(x-c_k)^{d_k}$ and dim $W_i = d_i, i = 1,...,k.$
3. dim W_1 + … dim W_k = dim V.

(OR)

(ii) Let T be a linear operator on a finite dimensional vector space V. If f is the characteristic polynomial for T, then Show that f(T) = 0.
(15)

II. a. i) Let T be a linear operator on a finite dimensional space V and let c be a scalar. Prove that the following statements are equivalent.

a) c is a characteristic value of T.
b) The operator (T – cI) is singular.
c) det (T- cI) = 0.

(OR)

ii) Let V be a finite dimensional vector space. Let $W_1, \dots, W_k$ be independent subspaces such that $W = W_1 + \dots + W_k$, then prove that $W_j \cap (W_1+...+W_{j-1}) = \{0\}$ for $2 \le j \le k.$ (5)

b. i) State and prove Primary Decomposition theorem.

(OR)

ii) Let T be linear operator on a finite dimensional space V and $c_1, \dots c_k$ be the distinct characteristic values of T. Prove that T is diagonalizable if and only if there exist k linear operators $E_1, \dots E_k$ on V such that

1. Each E_i is a projection.
2. $E_i E_j = 0, i \neq j.$
3. $I = E_1 + \dots + E_k$.
4. $T = c_1E_1 + \dots + c_k E_k$
5. The range of E_i is the characteristic space of T associated with c_i. (15)

III. a) i) Define T – admissible, T- annihilator, Projection of vector space V and Companion matrix.

(OR)

ii) Let T be a linear operator on a finite-dimensional vector space V. Let p and f be the minimal and characteristic polynomials for T, respectively.

(i) P divides f.

(ii) P and f have the same prime factors, except for multiplicities.

(iii) If $p = f_1^{T1} \dots f_k^{Tk}$

In the prime factorization of p, then $f = f_1^{d1} \dots f_k^{dk}$

where d_i is the nullity of $f_i\,(T)^{ri}$ divided by the degree of_i. (5)

b. i) State and prove Cyclic Decomposition Theorem.

(OR)

(ii) Let P be an m x n matrix with entries in the polynomial algebra F[x]. Show that following are equivalent.

1. P is invertible
2. The determinant of P is a non-zero scalar polynomial
3. P is row-equivalent to the m x n identity matrix
4. P is a product of elementary matrices. (15)

IV. a. i) Let V be a complex vector space and f be a form on V such that f(α, α) is real for every α. Then f is Hermitian.

(OR)

(ii) Define a positive matrix and verify that the matrix $\begin{pmatrix} 1 & 1+i \\ 1-i & 3 \end{pmatrix}$ is positive. (5)

b. i) Let V be a finite-dimensional inner product space and f a form on V. Then there is a unique linear operator T on V such that

$$f(\alpha, \beta) = (T\alpha|\beta)$$

for all α, β in V, and the map f→ T is an isomorphism of the space of forms onto L(V, V). (8)

(ii) For any linear operator T on a finite-dimensional inner product space V, there exists a unique linear T* on V such that

$$(T\alpha|\beta) = (\alpha|T^*\beta) \text{ for all } \alpha, \beta \text{ in V.}$$ (7)

(OR)

iii) Let W be a subspace of an inner product space V and let β be a vector in V. Then Prove:

1. The vector α in W is a best approximation to β by vectors in W if and only if β – α is orthogonal to every vector in W.

2. If a best approximation to β by vectors in W exists, it is unique.
3. If W is finite-dimensional and $\{\alpha_1, \dots, \alpha_n\}$ is any orthonormal basis for W, then the vector

$\alpha = \sum_k \frac{(\beta|\alpha_k)}{\|a_k\|^2}\alpha_k$ is the (unique) best approximation to β by vectors in W. (15)

V. a. i) Define: Bilinear forms, Bilinear function, Matrix of f in the ordered basis B, symmetric, Quadratic form, Skew Symmetric Bilinear form, Non – degenerate, Orthogonal matrix, Positive forms.

(OR)

ii) Let F be the field of real numbers or the field of complex numbers. Let A be an n x n matrix over F. Show that The function g defined by g (X, Y) = Y* AX is a positive form on the space F^{nx1} if an only if there exists and invertible n X n matrix P with entries in F such that A = P*P. (5)

b. i) Let V be an n-dimensional vector space over the field of real numbers, and let f be a symmetric bilinear form on V which has rank r. Then there is and ordered basis $\{\beta_1, \beta_2, \ldots, \beta_n\}$ for V in which the matrix of f is diagonal and such that

$$f(\beta_j, \beta_j) = \pm 1, \quad j = 1, \ldots, r.$$

Also show that the number of basis vectors β_j for which $f(\beta_j, \beta_j) = 1$ is independent of the choice of basis. (15)

(OR)

ii) Let V be a finite-dimensional vector space over the field of complex numbers. Let f be a symmetric bilinear form on V which has rank r. Then Prove that there is an ordered basis $B = \{\beta_1, \ldots, \beta_n\}$ for V such that

(i) the matrix of f in the ordered basis B is diagonal;

(ii) $f(\beta_j, \beta_j) = \begin{cases} 1, & j = 1, \ldots, r \\ 0, & j > r. \end{cases}$. (8)

ii) Let V be an inner product space and T a self – adjoint linear operator on V. Then prove that each characteristic value of T is real, and characteristic vectors of T associated with distinct characteristic value are orthogonal. (7)

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